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SELF-DUALITY AND RING EXTENSIONS

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Introduction

The purpose of this paper is to study the relation between ring extensions and self-duality. The theory of duality is established by K. Morita [2]. However, the problem concerning the existence of self-duality for a given ring is difficult and there seems to be only a few papers dealing with this problem. Indeed, the Artinian rings with self-duality have not yet been characterized. Recently K.R. Fuller and J.K. Haack [1] have proved that a finite 'zR semigroup ring' over a ring with self-duality has itself a self-duality.

In this paper, we shall generalize their result by establishing the following theorem.

Theorem. *Let $A \supseteq B$ be a ring extension. If A has a finite free basis over B each member of which centralizes all the elements of B, B has a self-duality induced by ${}_BV_B$, and if the structure constants of A with respect to the above basis commute with each element of V, then A has a self-duality induced by ${}_A\mathrm{Hom}_B({}_BA_A, {}_BV)$.*

It should be noted that any finite-dimensional algebra over a commutative field satisfies our condition, while it is not always a finite zR semigroup ring in general.

1.

Throughout this paper, $A \supseteq B$ is a ring extension satisfying the following conditions;

(1) A is a free left B-module of rank n.

(2) A has a free basis $x_1, \ldots, x_n \in A$ such that each x_i centralizes the elements of B, i.e.,

$$bx_i = x_i b \quad \text{for all } b \in B.$$

Under these assumptions, it is easy to check that A is a free right B-module of rank n. $Z(B)$ denotes the center of B, and put

$$x_i x_j = \sum_{p=1}^{n} \beta_{ij}^{p} x_p, \quad \beta_{ij}^{p} \in B,$$

$$1_A = \sum_{i=1}^{n} \alpha_i x_i, \qquad \alpha_i \in B.$$

Lemma 1.

(i) $\beta_{ij}^{p}, \alpha_i \in Z(B)$ *for all i, j and p.*

(ii) $\sum_p \beta_{ij}^{p} \beta_{pk}^{m} = \sum_p \beta_{jk}^{p} \beta_{ip}^{m}$ *for all i, j, k and m.*

(iii) $\sum_p \alpha_p \beta_{pj}^{i} = \delta_{ij} = \sum_p \alpha_p \beta_{jp}^{i}$ *for all i and j.*

Proof. (i) Since

$$\sum_p \beta_{ij}^{p} b x_p = \Big(\sum_p \beta_{ij}^{p} x_p\Big) b = (x_i x_j) b = b(x_i x_j)$$
$$= b\Big(\sum_p \beta_{ij}^{p} x_p\Big) = \sum_p b\beta_{ij}^{p} x_p \quad \text{for all } b \in B,$$

and $\{x_1, \dots, x_n\}$ is a free basis over B,

$$\beta_{ij}^{p} b = b\beta_{ij}^{p} \quad \text{for all } b \in B \text{ and } i, j \text{ and } p.$$

Next, since

$$\sum_i \alpha_i b x_i = \Big(\sum_i \alpha_i x_i\Big) b = 1 \cdot b = b \cdot 1$$
$$= b\Big(\sum_i \alpha_i x_i\Big) = \sum_i b\alpha_i x_i \quad \text{for all } b \in B,$$

we have

$$\alpha_i b = b\alpha_i \quad \text{for all } b \in B \text{ and } i.$$

(ii) This formula is well known.

(iii) Since

$$\sum_i \Big(\sum_p \alpha_p \beta_{pj}^{i}\Big) x_i = \sum_p \alpha_p \Big(\sum_i \beta_{pj}^{i} x_i\Big) = \sum_p \alpha_p x_p x_j$$
$$= \Big(\sum_p \alpha_p x_p\Big) x_j = x_j = \sum_i \delta_{ij} x_i \quad \text{for all } j,$$

we have

$$\sum_p \alpha_p \beta_{pj}^{i} = \delta_{ij} \quad \text{for all } i \text{ and } j. \qquad \square$$

From now on, assume that B has a self-duality induced by ${}_BV_B$. Then ${}_BV$ and V_B are the linearly compact injective cogenerators and

$$\mathrm{End}({}_BV)\cong B, \qquad \mathrm{End}(V_B)\cong B. \tag{3}$$

Put

$${}_BW_B=\bigoplus^n {}_BV_B.$$

We will think of each element of W as a row vector and denote $[v_s]$. For each $\sum_i b_ix_i\in A$ and $[v_s]\in W$, we define

$$\left(\sum_i b_ix_i\right)*[v_s]=\left[\sum_p\sum_i b_i\beta^p_{si}v_p\right]. \tag{4}$$

Lemma 2. *With the multiplication '$*$', W is a left A- right B-bimodule. Moreover, left B-module structure of $\bigoplus {}_BV$ coincides with the multiplication '$*$', i.e.,*

$$\left(\sum_i b\alpha_ix_i\right)*[v_s]=[bv_s] \quad \textit{for all } b\in B \textit{ and } [v_s]\in W.$$

Proof. We shall only prove

$$a*(a'*[v_s])=(aa')*[v_s] \tag{5}$$

for all $a,a'\in A$ and $[v_s]\in W$. Put $a=\sum_j b_jx_j$ and $a'=\sum_k b'_kx_k$. Then

$$\begin{aligned}a*(a'*[v_s])&=\left(\sum_j b_jx_j\right)*\left(\left(\sum_k b'_kx_k\right)*[v_s]\right)\\&=\left(\sum_j b_jx_j\right)*\left[\sum_q\sum_k b'_k\beta^q_{sk}v_q\right]=\left[\sum_p\sum_j\sum_q\sum_k b_j\beta^p_{sj}b'_k\beta^q_{pk}v_q\right].\end{aligned}$$

On the other hand,

$$\begin{aligned}(aa')*[v_s]&=\left(\left(\sum_j b_jx_j\right)\left(\sum_k b'_kx_k\right)\right)*[v_s]\\&=\left(\sum_p\sum_k\sum_j b_jb'_k\beta^p_{jk}x_p\right)*[v_s]=\left[\sum_q\sum_p\sum_k\sum_j b_jb'_k\beta^p_{jk}\beta^q_{sp}v_q\right].\end{aligned}$$

Then (5) follows from Lemma 1 (i) and (ii). $\square$

Hereafter, we will denote the multiplication omitting '$*$'.

Lemma 3.

$$\Phi:{}_A\mathrm{Hom}_B({}_BA_A,{}_BV_B)\ni\varphi\mapsto[(x_s)\varphi]\in{}_AW_B$$

is a bimodule isomorphism.

Proof. Since ${}_BA_B \cong \bigoplus^n {}_BB_B$, it is easy to show that Φ is a B-bimodule isomorphism. Let $\sum_k b_k x_k \in A$ and $\varphi \in \mathrm{Hom}_B({}_BA, {}_BV)$. Then

$$\begin{aligned}\left(\sum_k b_k x_k\right)[(x_s)\varphi] &= \left[\sum_p \sum_k b_k \beta_{sk}^p (x_p)\varphi\right] \\ &= \left[\left(\sum_p \sum_k b_k \beta_{sk}^p x_p\right)\varphi\right] = \left[\left(\sum_k b_k \sum_p \beta_{sk}^p x_p\right)\varphi\right] \\ &= \left[\left(\sum_k b_k x_s x_k\right)\varphi\right] = \left[(x_s)\left(\sum_k b_k x_k \varphi\right)\right].\end{aligned}$$

Hence

$$\left(\sum_k b_k x_k\right)\Phi(\varphi) = \Phi\left(\sum_k b_k x_k \varphi\right).$$

Thus we have proved that Φ is a A-homomorphism. □

Corollary 4. *${}_AW$ is a linearly compact injective cogenerator and defines a duality between A and* $\mathrm{End}({}_AW)$.

Proof. This corollary is directly from Lemma 3 and P. Vámos [3, Theorem 2.2]. □

Let $b \in Z(B)$. Since the map ${}_BV \ni v \mapsto bv \in {}_BV$ is a B-endomorphism of ${}_BV$, there uniquely exists $\pi(b) \in B$ such that

$$b'bv = b'v\pi(b) \quad \text{for all } v \in V \text{ and } b' \in B.$$

(Notice (3).) Thus we have

$$bv = v\pi(b) \quad \text{for all } v \in V.$$

Moreover,

$$\begin{aligned}v(\pi(b)b') &= (v\pi(b))b' = (bv)b' = b(vb') \\ &= (vb')\pi(b) = v(b'\pi(b)) \quad \text{for all } v \in V \text{ and } b' \in B.\end{aligned}$$

Since V_B is faithful, we have

$$\pi(b)b' = b'\pi(b) \quad \text{for all } b' \in B.$$

Therefore we have proved that $\pi(b) \in Z(B)$. Then

Lemma 5. *$\pi : Z(B) \to Z(B)$ is a ring automorphism.*

2.

We now proceed to compute $\mathrm{End}({}_AW)$. Since

$$\mathrm{End}({}_BW)=\mathrm{End}\left(\bigoplus^n {}_BV\right)=M_n(\mathrm{End}({}_BV))=M_n(B),$$

we have

$$B\subseteq \mathrm{End}({}_AW)\subseteq \mathrm{End}({}_BW)=M_n(B).$$

We will denote each element in $M_n(B)$ of the form (b_{pq}).

Lemma 6.

$$\sigma: A\ni \sum_i b_ix_i \mapsto \left(\sum_i b_i\beta^p_{iq}\right)\in M_n(B)$$

is a one-to-one ring homomorphism.

Proof. It is easy to show that σ is an additive homomorphism. First, we shall prove that σ is one-to-one.

$$\begin{aligned}
\sum_i b_ix_i\in\operatorname{Ker}\sigma \;&\Rightarrow\; \sum_i b_i\beta^p_{iq}=0 &&\text{for all } p \text{ and } q,\\
&\Rightarrow\; \sum_p\sum_i b_i\beta^p_{iq}x_p=0 &&\text{for all } q,\\
&\Rightarrow\; \sum_i b_ix_ix_q=0 &&\text{for all } q,\\
&\Rightarrow\; \sum_q\sum_i b_ix_ix_q\alpha_q=0,\\
&\Rightarrow\; \sum_i b_ix_i=0.
\end{aligned}$$

Hence σ is one-to-one. Next we shall prove that σ is a ring homomorphism. Let $\sum_i b_ix_i,\ \sum_j b'_jx_j\in A$. Then

$$\begin{aligned}
\sigma\left(\sum_i b_ix_i\right)\sigma\left(\sum_j b'_jx_j\right)&=\left(\sum_i b_i\beta^p_{iq}\right)\left(\sum_j b'_j\beta^p_{jq}\right)\\
&=\left(\sum_i\sum_j b_ib'_j\sum_t \beta^p_{it}\beta^t_{jq}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma\left(\left(\sum_i b_ix_i\right)\left(\sum_j b'_jx_j\right)\right)&=\sigma\left(\sum_i\sum_j b_ib'_j\sum_t \beta^t_{ij}x_t\right)\\
&=\left(\sum_i\sum_j b_ib'_j\sum_t \beta^t_{ij}\beta^p_{tq}\right).
\end{aligned}$$

Thus

$$\sigma\left(\sum_i b_ix_i\right)\sigma\left(\sum_j b'_jx_j\right)=\sigma\left(\left(\sum_i b_ix_i\right)\left(\sum_j b'_jx_j\right)\right)$$

by Lemma 1 (ii). $\square$

Corollary 7.

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & M_n(B) \\ \uparrow & & \uparrow \\ B & \hookrightarrow & \mathrm{End}({}_AW) \end{array}$$

is a commutative diagram. $\square$

Now we shall make the following assumption:

(C) $\pi(\beta_{ij}^k)=\beta_{ij}^k$ for all i,j and k.

(C) is equivalent to

$$\beta_{ij}^k v = v\beta_{ij}^k \quad \text{for all } v\in V \text{ and } i,j \text{ and } k.$$

Lemma 8. *Under the assumption* (C), *the followings are concluded.*

(i) *Let* $(b_{pq})\in M_n(B)$. *Then* $(b_{pq})\in \mathrm{End}({}_AW)$ *if and only if*

$$\sum_q \beta_{si}^q b_{tq} = \sum_q \beta_{qi}^t b_{qs} \quad \textit{for all } i,t \textit{ and } s.$$

(ii) $\mathrm{End}({}_AW)=\sum_{k=1}^n B(\beta_{kq}^p)=\sigma(A).$

Proof. (i) Let $[v_s]\in W$. Then

$$x_i([v_s](b_{pq}))=x_i\left[\sum_t v_t b_{ts}\right]=\left[\sum_t\sum_q \beta_{si}^q v_t b_{tq}\right]$$
$$=\left[\sum_t\sum_q v_t\beta_{si}^q b_{tq}\right] \quad \text{for all } i.$$

On the other hand,

$$(x_i[v_s])(b_{pq})=\left[\sum_t \beta_{si}^t v_t\right](b_{pq})=\left[\sum_t\sum_q \beta_{qi}^t v_t b_{qs}\right]$$
$$=\left[\sum_t\sum_q v_t\beta_{qi}^t b_{qs}\right] \quad \text{for all } i.$$

Thus we have

$$(b_{pq})\in \mathrm{End}({}_AW) \Leftrightarrow \sum_t\sum_q v_t\beta_{si}^q b_{tq}=\sum_t\sum_q v_t\beta_{qi}^t b_{qs}$$
$$\text{for all } [v_s]\in W \text{ and } i,s. \quad (6)$$

Suppose $(b_{pq})\in \mathrm{End}({}_AW)$. Let $v\in V$ and fix t. Put $[v_s]=[\delta_{ps}v]$. Then by (6),

$$v\sum_q \beta_{si}^q b_{tq}=v\sum_q \beta_{qi}^t b_{qs} \quad \text{for all } v\in V.$$

Therefore

$$\sum_q \beta^q_{si} b_{tq} = \sum_q \beta^t_{qi} b_{qs}.$$

Conversely, if $\sum_q \beta^q_{si} b_{tq} = \sum_q \beta^t_{qi} b_{qs}$ for all i and t, then it is easy to see that $(b_{pq}) \in \mathrm{End}({}_A W)$ from (6).

(ii) Let $(b_{pq}) \in \mathrm{End}({}_A W)$. Then, by (i),

$$b_{pi} = \sum_q \delta_{qi} b_{pq} = \sum_q \sum_s \alpha_s \beta^q_{si} b_{pq} = \sum_s \alpha_s \sum_q \beta^q_{si} b_{pq}$$

$$= \sum_s \alpha_s \sum_q \beta^p_{qi} b_{qs} = \sum_q \left(\sum_s \alpha_s b_{qs} \right) \beta^p_{qi} \quad \text{for all } p \text{ and } i.$$

Put $c_q = \sum_s \alpha_s b_{qs}$. Then we have

$$b_{pq} = \sum_k c_k \beta^p_{kq} \quad \text{for all } p \text{ and } q.$$

Thus

$$(b_{pq}) = \sum_k c_k (\beta^p_{kq}) \in \sum_k B(\beta^p_{kq}).$$

On the other hand, it is easy to check that $\sum_k B(\beta^p_{kq}) \subseteq \mathrm{End}({}_A W)$. Thus we have proved (ii). □

Now we get the following theorem.

Theorem 9. *Under the assumption* (C), *A has a self-duality induced by W.*

Proof. By Corollary 4, A has a duality induced by ${}_A W$, and $\mathrm{End}({}_A W) = \sigma(A) \cong A$ from Lemma 8. Thus ${}_A W$ induces a self-duality of A. □

References

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